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A GENERALIZATION OF A COMBINATORIAL IDENTITY WITH APPLICATIONS TO HIGHER BINOMIAL MOMENTS

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Abstract

A well-known combinatorial identity involving sums of integer powers is generalized. This generalization provides a new recurrence relation for the raw moments of the binomial distribution. Further, a similar recurrence relation for the central moments is derived. All these moments are recursively obtained from the corresponding moment of order zero, i.e., the unity.

1. Introduction

The calculation of the sum of the k-th powers (k positive integer) of the first n positive integers

$$S_k(n) = \sum_{j=1}^{n} j^k$$

has long interested mathematicians. For instance, the Bernoulli numbers B_n, usually defined by their exponential generating function

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$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \ |x| < 2\pi$$

were first studied by Jakob Bernoulli while computing the sums $S_k(n)$ [2]. In this context, the famous Faulhaber's formula (see, e.g., [7]), published in a 1631 edition of Academiae Algebrae,

$$S_k(n) = \frac{1}{k+1} \sum_{j=1}^{k+1} (-1)^{\delta_{jk}} \binom{k+1}{j} B_{k+1-j} n^j,$$

where δ_{jk} is the Kronecker delta, provides a closed form of the sum $S_k(n)$ as a polynomial in n of degree $k+1$, the coefficients involving Bernoulli numbers. A generalization of the Faulhaber's formula to complex powers with real part greater than -1 can be found in [9].

Regarding the positive integer case $(k \in \mathbb{N})$, among the many recent works dealing with the sums $S_k(n)$ (see, e.g., [3] and the references contained therein), in this paper, we focus our attention on the following recursion formula [5], which can be immediately derived using the binomial theorem

$$n^r = 1 + \sum_{k=0}^{r-1} \binom{r}{k} S_k(n-1) = 1 + \sum_{k=0}^{r-1} \binom{r}{k} \sum_{j=1}^{n-1} j^k. \tag{1.1}$$

Denoting $m_j^k := j^k (1 \leq j \leq n, \ 0 \leq k \leq r)$, Equation (1.1) can be rewritten as the recurrence relation

$$m_n^r = 1 + \sum_{k=0}^{r-1} \binom{r}{k} \sum_{j=1}^{n-1} m_j^k, \tag{1.2}$$

where $m_j^0 = j^0 = 1$ for all $j = 1, \ldots, n-1$.

With this reformulation (1.2) of Equation (1.1), we can consider its following natural generalization, obtained simply by multiplying the right-hand side of (1.2) by a real parameter $p(0 < p \leq 1)$, i.e.,

$$m_{n,p}^{r} = p\left\{1 + \sum_{k=0}^{r-1}\binom{r}{k}\sum_{j=1}^{n-1} m_{j,p}^{k}\right\}, \tag{1.3}$$

where $m_{j,p}^{0} = 1$ for all $j = 1, \ldots, n-1$ and for all $p \in (0, 1]$.

Of course, for $p = 1$ Equation (1.3) becomes Equation (1.2) (which is equivalent to (1.1)), but what happens with Equation (1.3) when $0 < p < 1$?

The main goal of this paper is to prove that for $0 < p < 1$, the value $m_{n,p}^{r}$ obtained by Equation (1.3) is the r-th raw moment (about the origin) in the binomial frequency distribution $B(n, p)$, for all $n \geq 2$, $r \geq 1$ and $0 < p < 1$. In this way, the recursion formula (1.1) for power sums is generalized to the new algebraic recursion formula (1.3) for binomial moments.

By the way, we obtain a recurrence relation that generalizes (from $p = 1/2$ to an arbitrary parameter $0 < p < 1$) the following recursive formula, derived in [1], for the raw moments of the binomial distribution $B(n, \frac{1}{2})$

$$m_{n,\frac{1}{2}}^{r} = n\left(m_{n,\frac{1}{2}}^{r-1} - \frac{1}{2} m_{n-1,\frac{1}{2}}^{r-1}\right). \tag{1.4}$$

Moreover, we also obtain a recursion formula, similar to Equation (1.3), for the central moments (about the mean np) of $B(n, p)$.

Throughout this paper we shall denote the r-th raw moment of $X \sim B(n, p)$ by

$$m_{n,p}^{r} := E[X^{r}] = \sum_{k=0}^{n} k^{r} p^{k}(1-p)^{n-k}\binom{n}{k} \text{ for all } r \geq 1,\ m_{0}^{n,p} = 1 \tag{1.5}$$

and the r-th central moment of $X \sim B(n, p)$ by

$$\mu_{n,p}^{r} := E[(X-np)^{r}] = \sum_{k=0}^{n}(k-np)^{r}p^{k}(1-p)^{n-k}\binom{n}{k} \text{ for all } r \geq 1,\ \mu_{0}^{n,p} = 1. \tag{1.6}$$

In particular, for the Bernoulli distribution $B(1, p)$, we have, for all $r \geq 1$

$$m_{1,p}^{r} = \sum_{k=0}^{1}k^{r}p^{k}(1-p)^{1-k}\binom{1}{k} = p, \tag{1.7}$$

$$\begin{aligned}\mu_{1,p}^{r} &= \sum_{k=0}^{1}(k-p)^{r}p^{k}(1-p)^{1-k}\binom{1}{k} = (-p)^{r}(1-p)+(1-p)^{r}p\\ &= p(1-p)\left[(-1)^{r}p^{r-1}+(1-p)^{r-1}\right].\end{aligned} \tag{1.8}$$

As is well-known, one can compute higher order moments about the origin of $X \sim B(n, p)$ using its characteristic function or its moment generating function; see, e.g., [8, 10]. A different combinatorial approach for obtaining higher order raw binomial moments that uses the factorial moments can be found in [6]. Let us denote by $X^{\underline{r}} = X(X-1)\cdots(X-r+1)$ the falling factorial and by $m_{n,p}^{\underline{r}} := E[X^{\underline{r}}]$ the r-th factorial moment of the binomial distribution $B(n, p)$. Then, in that work, the authors use the well-known expression for the Stirling numbers of the second kind $S(r, k)$ [4]

$$X^{r} = \sum_{k=1}^{r}S(r, k)X^{\underline{k}}$$

and take expectations in the both sides to get

$$m_{n,p}^{r} = \sum_{k=1}^{r}S(r, k)m_{n,p}^{\underline{k}} = \sum_{k=1}^{r}S(r, k)n^{\underline{k}}p^{k},$$

since the k-th factorial moment of the binomial distribution $B(n, p)$ is given by $m_{n,p}^{\underline{k}} = n^{\underline{k}}p^{k}$ [8].

Then, the central moments can be expressed in terms of the raw moments simply using the binomial theorem [10].

Alternatively, the combinatorial formulas presented here for obtaining the raw (central, respectively) moments of order r of $X \sim B(n, p)$ exclusively involve raw (central, respectively) moments of lower orders, and they will enable one to obtain the general expressions of all moments of $B(n, p)$ from the moment of order zero, i.e., $E[1] = 1$.

In Section 2, we present a simple recursive combinatorial formula from which all recurrence relations will be obtained. Sections 3 and 4 are, respectively, devoted to obtain the recurrence relations for the binomial raw and central moments. Finally, in Section 5, we present some closing remarks.

2. A Basic Lemma

The next lemma provides us with a simple recursive combinatorial formula, that will be used in the next two sections.

Lemma 2.1. *Let n be a positive integer and* $a_0, a_1, \ldots, a_n \in \mathbb{R}$. *Then*

$$\sum_{i=0}^{n} a_i \binom{n}{i} = \sum_{i=0}^{n-1} (a_i + a_{i+1}) \binom{n-1}{i}. \tag{2.1}$$

Proof. For $n = 1$, the formula is obvious. Otherwise, we have

$$\begin{aligned}
\sum_{i=0}^{n} a_i \binom{n}{i} &= a_0 \binom{n}{0} + \sum_{i=1}^{n-1} a_i \binom{n}{i} + a_n \binom{n}{n} \\
&= a_0 \binom{n-1}{0} + \sum_{i=1}^{n-1} a_i \left[\binom{n-1}{i} + \binom{n-1}{i-1} \right] + a_n \binom{n-1}{n-1} \\
&= \sum_{i=0}^{n-1} a_i \binom{n-1}{i} + \sum_{i=1}^{n} a_i \binom{n-1}{i-1} \\
&= \sum_{i=0}^{n-1} a_i \binom{n-1}{i} + \sum_{h=0}^{n-1} a_{h+1} \binom{n-1}{h} = \sum_{i=0}^{n-1} (a_i + a_{i+1}) \binom{n-1}{i}. \qquad \square
\end{aligned}$$

In the next two sections, we get some recurrence relations for the raw and central binomial moments, simply assigning the adequate values to the parameters a_k in Equation (2.1).

3. Raw Moments

In the next proposition, we establish a natural generalization of Equation (1.4).

Proposition 3.1. *Let* $n \geq 2$ *and* $0 < p < 1$. *Then for all* $r \geq 1$

$$m_{n,p}^{r} = n\left(m_{n,p}^{r-1} - (1-p)m_{n-1,p}^{r-1}\right).$$

Proof. Using formula (2.1) with $a_k = k^r p^k (1-p)^{n-k} (0 \leq k \leq n)$, we get

$$\begin{aligned}
m_{n,p}^{r} &= \sum_{k=0}^{n} k^r p^k (1-p)^{n-k} \binom{n}{k} \\
&= \sum_{k=0}^{n-1} \left[k^r p^k (1-p)^{n-k} + (k+1)^r p^{k+1} (1-p)^{n-(k+1)}\right] \binom{n-1}{k} \\
&= (1-p) \sum_{k=0}^{n-1} k^r p^k (1-p)^{(n-1)-k} \binom{n-1}{k} \\
&\quad + \sum_{k=0}^{n-1} (k+1)^r p^{k+1} (1-p)^{n-(k+1)} \frac{k+1}{n} \binom{n}{k+1} \\
&= (1-p) m_{n-1,p}^{r} + \frac{1}{n} \sum_{h=0}^{n} h^{r+1} p^h (1-p)^{n-h} \binom{n}{h} \\
&= (1-p) m_{n-1,p}^{r} + \frac{1}{n} m_{n,p}^{r+1},
\end{aligned}$$

and then

$$m_{n,p}^{r+1} = n\left(m_{n,p}^{r} - (1-p)m_{n-1,p}^{r}\right), \text{ i.e., } m_{n,p}^{r} = n\left(m_{n,p}^{r-1} - (1-p)m_{n-1,p}^{r-1}\right). \quad \square$$

For proving Equation (1.3), we need the following lemma. The proof of this lemma is analogous to the one of Proposition 3.1: We use again formula (2.1), but now with a different algebraic manipulation.

Lemma 3.1. *Let* $n \geq 2$ *and* $0 < p < 1$. *Then for all* $r \geq 1$

$$m_{n,p}^{r} = m_{n-1,p}^{r} + p\sum_{k=0}^{r-1}\binom{r}{k}m_{n-1,p}^{k}. \tag{3.1}$$

Proof. Using Equation (1.5) and formula (2.1) with $a_i = i^r p^i (1-p)^{n-i} (0 \leq i \leq n)$, we get

$$\begin{aligned}
m_{n,p}^{r} &= \sum_{i=0}^{n} i^r p^i (1-p)^{n-i}\binom{n}{i} \\
&= \sum_{i=0}^{n-1}\left[i^r p^i (1-p)^{n-i} + (i+1)^r p^{i+1}(1-p)^{n-(i+1)}\right]\binom{n-1}{i} \\
&= (1-p)\sum_{i=0}^{n-1} i^r p^i (1-p)^{(n-1)-i}\binom{n-1}{i} \\
&\quad + p\sum_{i=0}^{n-1}(i+1)^r p^i (1-p)^{(n-1)-i}\binom{n-1}{i} \\
&= (1-p)m_{n-1,p}^{r} + p\sum_{i=0}^{n-1}\left[1 + \sum_{k=1}^{r}\binom{r}{k}i^k\right]p^i(1-p)^{(n-1)-i}\binom{n-1}{i} \\
&= (1-p)m_{n-1,p}^{r} + p[p + (1-p)]^{n-1} \\
&\quad + p\sum_{k=1}^{r}\binom{r}{k}\sum_{i=0}^{n-1} i^k p^i (1-p)^{(n-1)-i}\binom{n-1}{i} \\
&= (1-p)m_{n-1,p}^{r} + p + p\sum_{k=1}^{r}\binom{r}{k}m_{n-1,p}^{k}
\end{aligned}$$

$$= (1-p)m_{n-1,p}^{r} + pm_{n-1,p}^{r} + p\sum_{k=0}^{r-1}\binom{r}{k}m_{n-1,p}^{k}$$

$$= m_{n-1,p}^{r} + p\sum_{k=0}^{r-1}\binom{r}{k}m_{n-1,p}^{k}. \qquad \square$$

Using Lemma 3.1, we immediately obtain Equation (1.3) in the next proposition.

Proposition 3.2. *Let* $n \geq 2$ *and* $0 < p < 1$. *Then for all* $r \geq 1$

$$m_{n,p}^{r} = p\left\{1 + \sum_{k=0}^{r-1}\binom{r}{k}\sum_{j=1}^{n-1}m_{j,p}^{k}\right\}.$$

Proof. Applying recursively formula (3.1) and using Equation (1.7), we get

$$m_{n,p}^{r} = m_{n-1,p}^{r} + p\sum_{k=0}^{r-1}\binom{r}{k}m_{n-1,p}^{k} = m_{n-2,p}^{r} + p\sum_{k=0}^{r-1}\binom{r}{k}\left(m_{n-2,p}^{k} + m_{n-1,p}^{k}\right)$$

$$= \cdots = m_{1,p}^{r} + p\sum_{k=0}^{r-1}\binom{r}{k}\left(m_{1,p}^{k} + \cdots + m_{n-1,p}^{k}\right)$$

$$= p + p\sum_{k=0}^{r-1}\binom{r}{k}\sum_{j=1}^{n-1}m_{j,p}^{k} = p\left\{1 + \sum_{k=0}^{r-1}\binom{r}{k}\sum_{j=1}^{n-1}m_{j,p}^{k}\right\}. \qquad \square$$

Example 3.1. Using Equation (1.3) and, as unique background, the obvious fact that $m_{j,p}^{0} = E[1] = 1$ for all $j \in \mathbb{N}$, we can recursively compute all raw moments, e.g.,

$$m_{n,p}^{1} = p\left\{1 + \sum_{i=0}^{0}\binom{1}{i}\sum_{j=1}^{n-1}m_{j,p}^{i}\right\} = p\left(1 + \sum_{j=1}^{n-1}m_{j,p}^{0}\right) = p\left(1 + \sum_{j=1}^{n-1}1\right) = np,$$

$$m_{n,p}^{2} = p\left\{1 + \sum_{i=0}^{1}\binom{2}{i}\sum_{j=1}^{n-1}m_{j,p}^{i}\right\} = p\left(1 + \sum_{j=1}^{n-1}m_{j,p}^{0} + 2\sum_{j=1}^{n-1}m_{j,p}^{1}\right)$$

$$= p\left(1 + \sum_{j=1}^{n-1} 1 + 2p\sum_{j=1}^{n-1} j\right) = n^2p^2 + np(1-p),$$

$$m_{n,p}^3 = p\left\{1 + \sum_{i=0}^{2}\binom{3}{i}\sum_{j=1}^{n-1} m_{j,p}^i\right\} = p\left(1 + \sum_{j=1}^{n-1} m_{j,p}^0 + 3\sum_{j=1}^{n-1} m_{j,p}^1 + 3\sum_{j=1}^{n-1} m_{j,p}^2\right)$$

$$= p\left(1 + \sum_{j=1}^{n-1} 1 + 3p\sum_{j=1}^{n-1} j + 3p^2\sum_{j=1}^{n-1} j^2 + 3p(1-p)\sum_{j=1}^{n-1} j\right)$$

$$= n^3p^3 + 3n^2p^2(1-p) + np(1-p)(1-2p).$$

4. Central Moments

The formula (1.3), for the raw moments, can be established for central moments $\mu_{n,p}^r$ of order r of $B(n, p)$. Now the proof is similar to the one used for Proposition 3.2, although a bit more complicated because the mean jp of $B(j, p)$ depends on the parameter j, $1 \le j \le n$.

Lemma 4.1. *Let* $n \ge 2$ *and* $0 < p < 1$. *Then for all* $r \ge 2$

$$\mu_{n,p}^r = \mu_{n-1,p}^r + p(1-p)\sum_{i=0,\, i\neq 1}^{r-2}\binom{r}{i} S_{i,r}\mu_{n-1,p}^i, \tag{4.1}$$

where $S_{i,r} := (-1)^{r-i} p^{r-i-1} + (1-p)^{r-i-1}$ *for all* $i = 0, 1, \ldots, r-1$.

Proof. Using Equation (1.6) and formula (2.1) with $a_k = (k-np)^r p^k(1-p)^{n-k} (0 \le k \le n)$, and denoting, for short, $\prod_k := p^k(1-p)^{(n-1)-k} \binom{n-1}{k}$ for every $k = 0, 1, \ldots, n-1$, we get

$$\mu_{n,p}^r = \sum_{k=0}^{n} (k-np)^r p^k (1-p)^{n-k}\binom{n}{k}$$

$$= \sum_{k=0}^{n-1}\left[(k-np)^r p^k(1-p)^{n-k} + (k+1-np)^r p^{k+1}(1-p)^{n-(k+1)}\right]\binom{n-1}{k}$$

$$= \sum_{k=0}^{n-1} \left[(k - np)^r (1 - p) + (k - np + 1)^r p \right] \Pi_k$$

$$= \sum_{k=0}^{n-1} \{ [(k - (n-1)p) - p]^r (1 - p) + [(k - (n-1)p) + (1 - p)]^r p \} \Pi_k$$

$$= \sum_{k=0}^{n-1} \sum_{i=0}^{r} \binom{r}{i} (k - (n-1)p)^i \left[(-p)^{r-i} (1 - p) + (1 - p)^{r-i} p \right] \Pi_k$$

$$= \sum_{k=0}^{n-1} (k - (n-1)p)^r \, \Pi_k$$

$$+ \sum_{k=0}^{n-1} \sum_{i=0}^{r-1} \binom{r}{i} (k - (n-1)p)^i \left[(-p)^{r-i} (1 - p) + (1 - p)^{r-i} p \right] \Pi_k$$

$$= \mu_{n-1,\,p}^{r}$$

$$+ \sum_{k=0}^{n-1} \sum_{i=0}^{r-1} \binom{r}{i} (k - (n-1)p)^i \left[(-p)^{r-i} (1 - p) + (1 - p)^{r-i} p \right] \Pi_k$$

$$= \mu_{n-1,\,p}^{r}$$

$$+ \sum_{i=0}^{r-1} \binom{r}{i} \left[(-p)^{r-i} (1 - p) + (1 - p)^{r-i} p \right] \sum_{k=0}^{n-1} (k - (n-1)p)^i \, \Pi_k$$

$$= \mu_{n-1,\,p}^{r} + \sum_{i=0}^{r-1} \binom{r}{i} \left[(-p)^{r-i} (1 - p) + (1 - p)^{r-i} p \right] \mu_{n-1,\,p}^{i}$$

$$= \mu_{n-1,\,p}^{r} + p(1 - p) \sum_{i=0}^{r-1} \binom{r}{i} \left[(-1)^{r-i} p^{r-i-1} + (1 - p)^{r-i-1} \right] \mu_{n-1,\,p}^{i}$$

$$= \mu_{n-1,\,p}^{r} + p(1 - p) \sum_{i=0}^{r-1} \binom{r}{i} S_{i,r} \mu_{n-1,\,p}^{i}$$

$$= \mu_{n-1,\,p}^{r} + p(1 - p) \sum_{i=0,\, i \neq 1}^{r-2} \binom{r}{i} S_{i,r} \mu_{n-1,\,p}^{i},$$

since, for $i = 1 : \mu^1_{n-1,p} = 0$, while for $i = r - 1 : S_{r-1,r} = 0$. □

In the next proposition, we present our recurrence relation for the central moments.

Proposition 4.1. *Let* $n \geq 2$ *and* $0 < p < 1$. *Then for all* $r \geq 2$

$$\mu^r_{n,p} = p(1-p)\left\{S_{0,r} + \sum_{i=0, i\neq 1}^{r-2} \binom{r}{i} S_{i,r} \sum_{j=1}^{n-1} \mu^i_{j,p}\right\}, \tag{4.2}$$

where $S_{i,r} := (-1)^{r-i} p^{r-i-1} + (1-p)^{r-i-1}$ *for all* $i = 0, 2, \ldots, r-2$.

Proof. Applying recursively formula (4.1) and using Equation (1.8), we get

$$\mu^r_{n,p} = \mu^r_{n-1,p} + p(1-p) \sum_{i=0, i\neq 1}^{r-2} \binom{r}{i} S_{i,r}\, \mu^i_{n-1,p}$$

$$= \mu^r_{n-2,p} + p(1-p) \sum_{i=0, i\neq 1}^{r-2} \binom{r}{i} S_{i,r} (\mu^i_{n-2,p} + \mu^i_{n-1,p})$$

$$= \cdots = \mu^r_{1,p} + p(1-p) \sum_{i=0, i\neq 1}^{r-2} \binom{r}{i} S_{i,r} (\mu^i_{1,p} + \cdots + \mu^i_{n-1,p})$$

$$= p(1-p)\left[(-1)^r p^{r-1} + (1-p)^{r-1}\right] + p(1-p) \sum_{i=0, i\neq 1}^{r-2} \binom{r}{i} S_{i,r} \sum_{j=1}^{n-1} \mu^i_{j,p}$$

$$= p(1-p)\left\{S_{0,r} + \sum_{i=0, i\neq 1}^{r-2} \binom{r}{i} S_{i,r} \sum_{j=1}^{n-1} \mu^i_{j,p}\right\}.$$ □

Example 4.1. Using Equation (4.2) and, as unique background, the obvious fact that $\mu^0_{j,p} = E[1] = 1$ for all $j \in \mathbb{N}$, we can recursively compute all central moments, e.g.,

$$\mu_{n,p}^{2} = p(1-p)\left\{S_{0,2} + \sum_{i=0,\, i\neq 1}^{0}\binom{2}{i}S_{i,2}\sum_{j=1}^{n-1}\mu_{j,p}^{i}\right\}$$

$$= p(1-p)\left(1 + \sum_{j=1}^{n-1}\mu_{j,p}^{0}\right) = p(1-p)\left(1 + \sum_{j=1}^{n-1}1\right) = np(1-p),$$

$$\mu_{n,p}^{3} = p(1-p)\left\{S_{0,3} + \sum_{i=0,\, i\neq 1}^{1}\binom{3}{i}S_{i,3}\sum_{j=1}^{n-1}\mu_{j,p}^{i}\right\}$$

$$= p(1-p)(1-2p)\left(1 + \sum_{j=1}^{n-1}\mu_{j,p}^{0}\right)$$

$$= p(1-p)(1-2p)\left(1 + \sum_{j=1}^{n-1}1\right) = np(1-p)(1-2p),$$

$$\mu_{n,p}^{4} = p(1-p)\left\{S_{0,4} + \sum_{i=0,\, i\neq 1}^{2}\binom{4}{i}S_{i,4}\sum_{j=1}^{n-1}\mu_{j,p}^{i}\right\}$$

$$= p(1-p)\left((3p^{2}-3p+1)\left(1 + \sum_{j=1}^{n-1}\mu_{j,p}^{0}\right) + 6\sum_{j=1}^{n-1}\mu_{j,p}^{2}\right)$$

$$= p(1-p)\left((3p^{2}-3p+1)\left(1 + \sum_{j=1}^{n-1}1\right) + 6p(1-p)\sum_{j=1}^{n-1}j\right)$$

$$= np(1-p)[1 + 3(n-2)p(1-p)].$$

5. Closing Remarks

A natural generalization of the binomial identity (1.1) led to a new recurrence relation for the raw moments of the binomial distribution (Propositions 3.2). A similar expression was derived for the central moments (Proposition 4.1). The following properties of such two expressions can be highlighted.

Remark 5.1. Equations (1.3) and (4.2) enable one to obtain all raw and central binomial moments, respectively, of any order r, simply starting from the moment of order zero, i.e., the unity. Moreover, for every given parameters, n, p, the recursive character of these formulas naturally leads to easily implementable algorithms for rapidly computing all (raw and central) moments of $B(n, p)$.

Remark 5.2. The formulas (3.1) and (1.3), established for the raw moments, can be generalized for general moments of order r of $B(n, p)$ about an arbitrary point a, denoted by $M_{n,p}^{r}(a)$. More precisely, taking into account that

$$m_{j,p}^{i} = M_{j,p}^{i}(0) \quad (0 \leq i \leq r, 1 \leq j \leq n),$$

then simply writing $M_{j,p}^{i}(a)$ instead of $m_{j,p}^{i}$ all along the proofs of Lemma 3.1 and Proposition 3.2, we get the following generalizations of Equations (3.1) and (1.3)

$$M_{n,p}^{r}(a) = M_{n-1,p}^{r}(a) + p\sum_{i=0}^{r-1}\binom{r}{i}M_{n-1,p}^{i}(a),$$

$$M_{n,p}^{r}(a) = p\left\{1 + \sum_{i=0}^{r-1}\binom{r}{i}\sum_{j=1}^{n-1}M_{j,p}^{i}(a)\right\}.$$

Remark 5.3. As Example 3.1 shows, the recursion formula (1.3) expresses the raw binomial moments $m_{n,p}^{r}$ in terms of the power sums $S_k(n-1)$, in a similar way as Equation (1.1)-our starting point-expresses n^r in terms of those power sums, e.g.,

$$m_{n,p}^{1} = p\left(1 + \binom{1}{0}S_0(n-1)\right),$$

$$m_{n,p}^{2} = p\left(1 + \binom{2}{0}S_0(n-1) + \binom{2}{1}pS_1(n-1)\right),$$

$$m_{n,p}^{3} = p\left(1 + \binom{3}{0}S_0(n-1) + \binom{3}{1}p(2-p)S_1(n-1) + \binom{3}{2}p^2S_2(n-1)\right).$$

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